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# A restriction theorem for Métivier groups

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**Abstract**

In the spirit of an earlier result of D. Müller on the Heisenberg group we prove a restriction theorem on a certain class of two step nilpotent Lie groups. Our result extends that of Müller also in the framework of the Heisenberg group.



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*Keywords:* Restriction theorem; Two-step nilpotent groups; Métivier groups; SubLaplacian; Twisted Laplacian

## 1. Introduction

In this paper we examine the mapping properties between Lebesgue spaces of the operators arising in the spectral resolution of the subLaplacian on the class of groups introduced by G. Métivier in [6]. These are two-step nilpotent Lie groups, characterized by the property that the quotients with respect to the hyperplanes contained in the centre are Heisenberg groups. The groups of $H$-type, introduced by A. Kaplan [4], are examples of groups satisfying the Métivier property, but there are Métivier groups which are not of $H$-type (for an example, see [9]).

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Let $G$ be a Métivier group equipped with a subLaplacian $L$. The operators $\mathcal{P}_\mu^L$, we are interested in, are formally given by a Dirac delta $\delta_\mu(L)$ at a point $\mu$ of the spectrum of $L$. $\mathcal{P}_\mu^L$ corresponds to a "generalized projection operator" in the sense introduced by R. Strichartz in [15]: its range consists of eigenfunctions of $L$, and the family $\{\mathcal{P}_\mu^L\}$ decomposes $f$, in the sense that for all Schwartz functions $f$ on $G$

$$f = \int_0^{+\infty} \mathcal{P}_\mu^L f \, d\mu,$$

where the integral converges in the sense of distributions, as we shall prove in Theorem 4.8.

On the Euclidean space $\mathbb{R}^d$ the spectral resolution of the Laplacian $\Delta = -\partial_1^2 - \cdots - \partial_d^2$ may be given in terms of convolutions with the Fourier transform of the measures $d\sigma_r$, induced on the spheres centred at the origin by the Lebesgue measure, since $\Delta f * \widehat{\sigma_r} = r^2 f * \widehat{\sigma_r}$. The celebrated Stein–Tomas theorem [13, Chapter 9] describes the mapping properties of the convolution operator with $\widehat{\sigma_r}$. Throughout the paper, we adopt the following notation

$$p_*(d) := 2\frac{d+1}{d+3}, \quad d \in \mathbb{N}. \tag{1.1}$$

**Theorem 1.1** (*Stein–Tomas Restriction Inequality*). *Suppose that $1 \le p \le p_*(d)$ and let $\frac{1}{p} + \frac{1}{p'} = 1$. Then the estimate*

$$\|f * \widehat{\sigma_r}\|_{p'} \le C_r \|f\|_p \tag{1.2}$$

*holds for all Schwartz functions on $\mathbb{R}^d$ and all $r > 0$.*

According to the Knapp example [13], estimate (1.2) fails if $p > p_*$.

Strichartz suggested to study the boundedness properties of the operators arising in the spectral resolution of other Laplacians. D. Müller, motivated also by the works of C. Sogge on the spectral projections of the Laplace–Beltrami operator on compact Riemannian manifolds [11,12], proved an analogue of the Stein–Tomas theorem for the subLaplacian on the Heisenberg group [8].

**Theorem 1.2** (*D. Müller*). *Let $1 \le p \le 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$. The inequality*

$$\|\mathcal{P}_\mu^L f\|_{L_t^\infty L_z^{p'}} \le C_\mu \|f\|_{L_t^1 L_z^p}, \tag{1.3}$$

*holds for all Schwartz functions on $\mathbb{H}^n$ and all $\rho > 0$.*

The theorem is stated in terms of the mixed Lebesgue norms

$$\|f\|_{L_t^r L_z^p} = \left(\int_{\mathbb{C}^n} \left(\int_{-\infty}^{\infty} |f(z,t)|^r dt\right)^{\frac{p}{r}} dz\right)^{\frac{1}{p}}, \quad 1 \le p, r < \infty, \tag{1.4}$$

(with the obvious modifications when $p$ or $r$ is equal to $\infty$), since, as shown by Müller, the only available estimate between $L^p$ spaces on $\mathbb{H}^n$ is the trivial $L^1 - L^\infty$ one. In addition, a counterexample produced in [8] shows that $\mathcal{P}_\mu^L$ is unbounded as an operator between $L_t^r L_z^p$ and $L_t^{r'} L_z^{p'}$, unless $r = 1$. The main reason for that is that the operators $\mathcal{P}_\mu^L$ operate on the $t$ variable through the Fourier transform, but the Heisenberg group has one dimensional centre and there are no nontrivial restriction estimates for the Fourier transform on the real line. Indeed, S. Thangavelu proved in [17] that the inequality

$$\|\mathcal{P}_\mu^L f\|_{L^{p'}(G)} \leq C\|f\|_{L^p(G)}$$

holds for $1 \leq p \leq p_*(n)$ on the direct product $G$ of $n$ copies of the three dimensional Heisenberg group $\mathbb{H}_1$.

We extend Müller's theorem in two ways. First, since the dimension of the centre of Métivier groups is in general bigger than one (actually, in the Métivier class only the Heisenberg groups have a one dimensional centre), we incorporate the Stein–Tomas theorem in the estimate concerning the central variables. Second, we improve (1.3) by replacing on the left-hand side $p'$ with an exponent $q < p'$. More precisely, we will prove the following result.

**Theorem 1.3.** *Let $G$ be a Métivier group, with Lie algebra $\mathfrak{g}$. Let $\mathfrak{z}$ and $\mathfrak{v}$ denote, respectively, the centre of $\mathfrak{g}$ and its orthogonal complement.*

*If $\dim\mathfrak{z} = d$ and $\dim\mathfrak{v} = 2n$, and if $1 \leq r \leq p_*(d)$, then for all $p, q$ satisfying $1 \leq p \leq 2 \leq q \leq \infty$ and for all Schwartz functions $f$, we have*

$$\|\mathcal{P}_\mu^L f\|_{L^{r'}(\mathfrak{z})L^q(\mathfrak{v})} \leq C\mu^{d\left(\frac{2}{r}-1\right)+n\left(\frac{1}{p}-\frac{1}{q}\right)-1}\|f\|_{L^r(\mathfrak{z})L^p(\mathfrak{v})}, \quad \mu > 0. \tag{1.5}$$

Here $C$ is independent of $f$ and $\mu$ and the definition of the norms is analogous to that given in the Heisenberg framework.

To explain the strategy, we recall that the operators $\mathcal{P}_\mu^L$ are given by the action in $z$ of the spectral projections of the twisted Laplacian, conjugated with the Fourier transform in the central variable. The twisted Laplacian is a second order elliptic differential operator on $\mathbb{C}^n$ with point spectrum. The estimates of the norms between Lebesgue spaces of its spectral projections, which are an essential ingredient in the proof of (1.5), have been progressively strengthened in the last twenty years (see for instance [10,14]); eventually the sharp $L^p - L^2$ bounds have been attained by H. Koch and F. Ricci [5] (see also [1] for a different proof). By incorporating in our argument the optimal estimates we obtain a result, which improves on that of Müller also on the Heisenberg group (in fact, for a Schwartz function $f$ on $\mathbb{H}^n$ we prove that

$$\|\mathcal{P}_\mu^L f\|_{L_t^\infty L_z^2} \leq C_\mu\|f\|_{L_t^1 L_z^p},$$

for all $1 \leq p \leq 2$).

It is worth noticing that from the estimates for the operators $\mathcal{P}_\mu^L$ one can deduce estimates for the standard spectral projections of the subLaplacian, which could be used to prove $L^p$ summability results for Bochner–Riesz means associated to the subLaplacian (see [7] for the Heisenberg case). We shall address this problem in the framework of Métivier groups in a forthcoming paper [3].

The paper is organized as follows. In Section 2 we recall the spectral resolution of the subLaplacian on the Heisenberg group, and use the Koch–Ricci estimates for the twisted Laplacian to strengthen Müller's estimate. In Section 3 we present some restriction estimates for the full Laplacian (defined by (2.1)) on the Heisenberg group. In Section 4 we compute the spectral resolution of the subLaplacian on a Métivier group $G$. Following a well known procedure (see [15,16]), by taking the Radon transform in the central variables and using the Métivier property, we reduce the computation of the spectral decomposition of a function on $G$ to the spectral decomposition of its Radon transform on a Heisenberg group. In Section 5 we prove the restriction theorem on the Métivier groups. An essential tool is given by a conditional statement, based on the assumption that the spectral projections of the twisted Laplacian are bounded between two Lebesgue spaces. We conclude by showing that the range of $r$ in (1.5) is sharp.

## 2. Restriction estimates for the subLaplacian on the Heisenberg group

In this section we discuss the case of the Heisenberg group. The Heisenberg group $\mathbb{H}_n$ is the space $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ equipped with the product

$$(x, y, t)(x', y', t') = \left(x + x', y + y', t + t' + \frac{1}{2}(x \cdot y' - x' \cdot y)\right),$$

for $x, x', y, y'$ in $\mathbb{R}^n$ and $t, t'$ in $\mathbb{R}$. This product turns $\mathbb{H}_n$ into a two step nilpotent Lie group with centre given by $\{(0, 0, t) : t \in \mathbb{R}\}$.

The algebra, $\mathfrak{h}_n$, of left invariant vector fields on $\mathbb{H}_n$ is spanned by

$$X_j = \frac{\partial}{\partial x_j} - \frac{1}{2} y_j \frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} + \frac{1}{2} x_j \frac{\partial}{\partial t}, \qquad T = \frac{\partial}{\partial t},$$

$j = 1, \ldots, n$. In terms of these vector fields we introduce on $\mathbb{H}_n$ the subLaplacian

$$L = -\sum_{j=1}^{n} \left(X_j^2 + Y_j^2\right),$$

which is hypoelliptic since the set $\{X_1 \ldots, Y_n\}$ generates $\mathfrak{h}_n$ as a Lie algebra, and the full Laplacian

$$\Delta_{\mathbb{H}} = -\sum_{j=1}^{n} \left(X_j^2 + Y_j^2\right) - T^2 = L - T^2. \tag{2.1}$$

We will use complex coordinates

$$z_j = x_j + i y_j \quad \text{and} \quad \bar{z}_j = x_j - i y_j,$$

$j = 1, \ldots, n$. In these coordinates the Haar measure coincides with the Lebesgue measure $dzdt = dxdydt$.

The operators $L$ and $-iT$ extend to a pair of strongly commuting self-adjoint operators on $L^2(\mathbb{H}^n)$. They therefore admit a joint spectral decomposition, that we now briefly recall for the sake of completeness. For more details we refer the reader to the book [18].

Given a nonzero real number $\lambda$ and a point $(z, t) = (x + iy, t)$ in $\mathbb{H}_n$, we denote by $\pi_\lambda(z, t)$ the operator acting on $L^2(\mathbb{R}^n)$ defined by

$$\pi_\lambda(z, t)\phi(\xi) = e^{i\lambda t}\pi_\lambda(z)\phi(\xi) = e^{i\lambda\left(t + x\cdot\xi + \frac{1}{2}x\cdot y\right)}\phi(\xi + y),$$

where $\pi_\lambda(z) = \pi_\lambda(z, 0)$, so that $\pi_\lambda(z, t) = e^{i\lambda t}\pi_\lambda(z, 0) = e^{i\lambda t}\pi_\lambda(z)$. For each $\lambda \neq 0$ the map $\pi_\lambda$ from $\mathbb{H}_n$ to the group of unitary operators on $L^2(\mathbb{R}^n)$ is an irreducible representation of $\mathbb{H}_n$. These maps are called Schrödinger's representations. In terms of it we define the group Fourier transform of a Schwartz function $f$ on $\mathbb{H}_n$, which is given by

$$\mathbb{R} \setminus \{0\} \ni \lambda \mapsto \pi_\lambda(f) = \int_{\mathbb{H}_n} f(z, t)\pi_\lambda(z, t)\,dtdz = \int_{\mathbb{C}_n} f^{(\lambda)}(z)\pi_\lambda(z)\,dz,$$

where

$$f^{(\lambda)}(z) = \int_{-\infty}^{\infty} f(z,t)e^{i\lambda t}\,dt$$

is the ordinary Fourier transform of $f(z,t)$ in the central variable $t$. The information provided by $\pi_\lambda(f)$ suffices to reconstruct $f$, which is in fact given by

$$f(z,t) = \frac{1}{(2\pi)^{n+1}} \int_{-\infty}^{\infty} e^{-i\lambda t}\,\mathrm{tr}\left(\pi_\lambda(z)^*\pi_\lambda(f)\right)|\lambda|^n d\lambda, \tag{2.2}$$

where we denote by $\pi_\lambda(z)^*$ the adjoint of $\pi_\lambda(z)$ and by tr the trace of an operator on $L^2(\mathbb{R}^n)$.

The differential $d\pi_\lambda$ of $\pi_\lambda$ yields a representation of the Lie algebra $\mathfrak{h}_n$. This representation extends to a representation of the universal enveloping algebra denoted by the same symbol $d\pi_\lambda$. The derivative with respect to the central variable, $T$, is represented in this picture by the multiplication by $i\lambda$. The subLaplacian $L$ is represented by the rescaled Hermite operator

$$d\pi_\lambda(L) = \Delta + \lambda^2|\xi|^2,$$

where $\Delta = -\partial_1^2 - \cdots - \partial_n^2$ and $|\xi|^2 = \xi_1^2 + \cdots + \xi_n^2$.

The Hermite operator $d\pi_\lambda(L)$ has a pure point spectrum with eigenvalues $|\lambda|(2k+n)$ for $k = 0, 1, \ldots$. The eigenspace corresponding to the eigenvalue $|\lambda|(2k+n)$ has an orthonormal basis given by

$$\{\Phi_\alpha^{(\lambda)} : |\alpha| = k\},$$

where the functions $\Phi_\alpha^{(\lambda)}$ are defined, for each multiindex $\alpha = (\alpha_1, \ldots, \alpha_n)$ in $\mathbb{Z}_+^n$ of length $|\alpha| = \alpha_1 + \cdots + \alpha_n = k$, by

$$\Phi_\alpha^{(\lambda)}(\xi_1, \ldots, \xi_n) = |\lambda|^{\frac{n}{4}} h_{\alpha_1}(\sqrt{|\lambda|}\xi_1)\cdots h_{\alpha_n}(\sqrt{|\lambda|}\xi_n)$$

and the functions $h_i(t)$ are normalized one-dimensional Hermite functions in $L^2(\mathbb{R}, dt)$.

Therefore, $d\pi_\lambda(L)$ is represented by

$$d\pi_\lambda(L) = \sum_{k=0}^{\infty} |\lambda|(2k+n)P_k^\lambda, \tag{2.3}$$

where $P_k^\lambda : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is the projection onto the eigenspace corresponding to $|\lambda|(2k+n)$.

Inserting in (2.2) the decomposition $\sum_{k=0}^{\infty} P_k^\lambda$ of the identity operator on $L^2(\mathbb{R}^n)$, we obtain

$$f(z,t) = \frac{1}{(2\pi)^{n+1}} \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} e^{-i\lambda t}\,\mathrm{tr}\left(\pi_\lambda(z)^*\pi_\lambda(f)P_k^\lambda\right)|\lambda|^n d\lambda. \tag{2.4}$$

This decomposition may be thought of as the expansion of $f$ in joint eigenfunctions, $e^{-i\lambda t}\,\mathrm{tr}\,(\pi_\lambda(z)^*\pi_\lambda(f)P_k^\lambda)$, of $-iT$ and $L$.

To obtain a more explicit form for (2.4) we compute the trace of the operators $\pi_\lambda(z)^*\pi_\lambda(f)P_k^\lambda$, which is given by

$$\mathrm{tr}\left(\pi_\lambda(z)^*\pi_\lambda(f)P_k^\lambda\right) = \sum_{|\alpha|=k} \left(\Phi_\alpha^\lambda, \pi_\lambda(z)^*\pi_\lambda(f)\Phi_\alpha^\lambda\right)_{L^2(\mathbb{R}^n)}.$$

The sum may be expressed in a closed form, introducing the $\lambda$-twisted convolution on $\mathbb{C}^n$.

**Definition 2.1.** Let $\lambda$ be a nonzero real number. The $\lambda$-twisted convolution $h \times_\lambda g$ of two Schwartz functions $h, g$ on $\mathbb{C}^n$ is defined by

$$(h \times_\lambda g)(z) = \int_{\mathbb{C}^n} h(z-w)g(w)e^{\frac{i}{2}\lambda \Im m z\cdot\bar{w}}dw. \tag{2.5}$$

When $\lambda = 1$ we shall write $\times$ instead of $\times_1$.

Then

$$\sum_{|\alpha|=k} \left(\Phi_\alpha^\lambda, \pi_\lambda(z)^*\pi_\lambda(f)\Phi_\alpha^\lambda\right)_{L^2(\mathbb{R}^n)} = \frac{1}{(2\pi)^n} f^{(\lambda)} \times_\lambda \varphi_k^\lambda(z)$$

(see [18]), where

$$\varphi_k^\lambda(z) = \sum_{|\alpha|=k} (\pi_\lambda(z)\Phi_\alpha^{(\lambda)}, \Phi_\alpha^{(\lambda)})_{L^2(\mathbb{R}^n)} = \tilde{\varphi}_k(\sqrt{|\lambda|}|z|),$$

and $\tilde{\varphi}_k$ is the $k$th-Laguerre function normalized by $\|\tilde{\varphi}_k\|_{L^2(\mathbb{R}_+,t^{n-1}dt)} = 1$. If $g$ is a Schwartz function on $\mathbb{C}^n$ and $\lambda \neq 0$, we set

$$\Lambda_k^\lambda g(z) = \frac{1}{(2\pi)^n} g \times_\lambda \varphi_k^{|\lambda|}(z), \tag{2.6}$$

writing

$$\operatorname{tr}\left(\pi_\lambda(z)^*\pi_\lambda(f)P_k^\lambda\right) = \Lambda_k^\lambda f^{(\lambda)}(z).$$

The operators $\Lambda_k^\lambda$ are orthogonal projections in $L^2(\mathbb{C}^n, |\lambda|^n dz)$, since

$$\varphi_j \times \varphi_j = (2\pi)^n \varphi_j \quad \text{and} \quad \varphi_j \times \varphi_k = 0 \quad \text{if } j \neq k$$

(see [18, (1.4.30)]), which imply

$$|\lambda|^{2n}\varphi_j^{|\lambda|} \times \varphi_k^{|\lambda|} = (2\pi)^n \delta_{jk}|\lambda|^n \varphi_j^{|\lambda|}.$$

Then (2.4) takes the form

$$f(z,t) = \frac{1}{(2\pi)^{n+1}} \sum_{k=0}^\infty \int_{-\infty}^\infty e^{-i\lambda t}\, \Lambda_k^\lambda f^{(\lambda)}(z)|\lambda|^n d\lambda.$$

This decomposition together with the allied Plancherel formula

$$\int_{\mathbb{C}^n}\int_{-\infty}^\infty |f(z,t)|^2 dt dz = \frac{1}{(2\pi)^{2n+1}} \sum_{k=0}^\infty \int_{-\infty}^\infty \left\|\Lambda_k^\lambda f^{(\lambda)}\right\|^2_{L^2(\mathbb{C}^n)} |\lambda|^{2n} d\lambda, \tag{2.7}$$

is the starting point for the development of the joint functional calculus of $L$ and $T$. Indeed, given a bounded function $m : \mathbb{R}_+ \times \mathbb{R} \setminus \{0\} \to \mathbb{C}$, we define for a Schwartz function $f$

$$m(L,-iT)f(z,t) = \frac{1}{(2\pi)^{n+1}} \sum_{k=0}^\infty \int_{-\infty}^\infty m(|\lambda|(2k+n),\lambda)e^{-i\lambda t}\, \Lambda_k^\lambda f^{(\lambda)}(z)|\lambda|^n d\lambda. \tag{2.8}$$

Then by (2.7) we have

$$\int_{\mathbb{C}^n}\int_{-\infty}^\infty |m(L,-iT)f(z,t)|^2 dt dz \le \|m\|^2_{L^\infty(\mathbb{R}_+\times\mathbb{R}\setminus\{0\})} \int_{\mathbb{C}^n}\int_{-\infty}^\infty |f(z,t)|^2 dt dz.$$

We shall use (2.8) to introduce the operators $\delta_\mu(L)$ and $\delta_\mu(\Delta_{\mathbb{H}})$ for $\mu > 0$, which are defined for a Schwartz function $f$ by

$$\delta_\mu(D)f = \lim_{\epsilon\to 0+} \frac{1}{2\epsilon}\chi_{(\mu-\epsilon,\mu+\epsilon)}(D)f,$$

with $D = L, \Delta_{\mathbb{H}}$, where $\chi_{(\mu-\epsilon,\mu+\epsilon)}$ is the characteristic function of the interval $(\mu-\epsilon, \mu+\epsilon)$.

More generally, with the same techniques one can also consider operators of the form $\delta_\mu\,(m\,(L, -iT))$ for a suitable function $m$. In the following, $\lambda$ refers to the spectrum of $-iT$, and $|\lambda|(2k+n)$ to the spectrum of $L$. If $m$ is an even function of $\lambda$, we may rewrite (2.8) as

$$\begin{aligned} m(L, -iT)f(z,t) = \frac{1}{(2\pi)^{n+1}} \sum_{k=0}^{\infty} \int_0^\infty & m(\lambda(2k+n), \lambda) \\ & \times \Big(e^{-i\lambda t}\,\Lambda_k^\lambda f^{(\lambda)}(z) + e^{i\lambda t}\,\Lambda_k^{-\lambda} f^{(-\lambda)}(z)\Big)\lambda^n d\lambda. \end{aligned}$$

We also assume that $m(\lambda(2k+n), \lambda)$ is a differentiable function of $\lambda$ on $\mathbb{R}_+$, with strictly positive derivative, satisfying $\lim_{\lambda\to 0+} m(\lambda(2k+n), \lambda) = 0$ and $\lim_{\lambda\to+\infty} m(\lambda(2k+n), \lambda) = +\infty$. Then the equation $m(\lambda(2k+n), \lambda) = \mu$ may be solved for each $k$ to give $\lambda = \lambda_k^m(\mu)$. For notational simplicity, we shall write $\lambda_k(\mu)$ instead of $\lambda_k^m(\mu)$ and denote by $\lambda_k'$ the derivative of $\lambda_k$.

Replacing in the integral $\lambda$ with $\mu$, we obtain

$$\begin{aligned} m(L, -iT)f(z,t) = \frac{1}{(2\pi)^{n+1}} \sum_{k=0}^{\infty} \int_0^\infty & \mu\Big(e^{-i\lambda_k(\mu)t}\,\Lambda_k^{\lambda_k(\mu)} f^{(\lambda_k(\mu))}(z) \\ & + e^{i\lambda_k(\mu)t}\,\Lambda_k^{-\lambda_k(\mu)} f^{(-\lambda_k(\mu))}(z)\Big)\lambda_k(\mu)^n\lambda_k'(\mu)d\mu, \end{aligned}$$

which is the spectral decomposition of $m(L, -iT)$. Hence, the spectral resolution with respect to $m(L, -iT)$ of a Schwartz function $f$ is

$$\begin{aligned} f(z,t) = \frac{1}{(2\pi)^{n+1}} \sum_{k=0}^{\infty} \int_0^\infty & \Big(e^{-i\lambda_k(\mu)t}\,\Lambda_k^{\lambda_k(\mu)} f^{(\lambda_k(\mu))}(z) \\ & + e^{i\lambda_k(\mu)t}\,\Lambda_k^{-\lambda_k(\mu)} f^{(-\lambda_k(\mu))}(z)\Big)\lambda_k(\mu)^n\lambda_k'(\mu)d\mu. \end{aligned} \tag{2.9}$$

Given a Schwartz function $f$, its spectral resolution is given in terms of the distributions

$$\mathcal{P}_\mu^m f = \delta_\mu(m(L, -iT))f = \lim_{\varepsilon\to 0+} \frac{1}{2\varepsilon}\chi_{(\mu-\varepsilon,\mu+\varepsilon)}(m(L, -iT))f.$$

Since $\operatorname{tr}\big(\pi_\mu(z,t)^*\pi_\mu(f)\big)$ is a continuous function of $\mu$, this limit exists and is given by

$$\begin{aligned} \mathcal{P}_\mu^m f(z,t) = \sum_{k=0}^{\infty} \frac{\lambda_k(\mu)^n\lambda_k'(\mu)}{(2\pi)^{n+1}} & \Big(e^{-i\lambda_k(\mu)t}\,\Lambda_k^{\lambda_k(\mu)} f^{(\lambda_k(\mu))}(z) \\ & + e^{i\lambda_k(\mu)t}\,\Lambda_k^{-\lambda_k(\mu)} f^{(-\lambda_k(\mu))}(z)\Big). \end{aligned} \tag{2.10}$$

The inversion formula (2.9) may be written as

$$f(z,t) = \int_0^\infty \mathcal{P}_\mu^m f(z,t)d\mu,$$

where the integral converges in the sense of distributions.

In the case of the subLaplacian $m(\mu,\lambda)=\mu$, thus we have $\mu=|\lambda|(2k+n)$, which yields $\lambda_k(\mu)=\mu/2k+n$. For notational simplicity, we shall write $\mu_k$ instead of $\lambda_k(\mu)$.

Therefore,

$$\mathcal{P}^L_\mu f(z,t)=\frac{\mu^n}{(2\pi)^{n+1}}\sum_{k=0}^\infty\frac{1}{(2k+n)^{n+1}}\Big(e^{-i\mu_k t}\,\Lambda_k^{\mu_k}f^{(\mu_k)}(z)$$
$$+e^{i(\mu_k)t}\,\Lambda_k^{-\mu_k}f^{(-\mu_k)}(z)\Big). \tag{2.11}$$

In the case of the full Laplacian $m(\mu,\lambda)=\mu+\lambda^2$, hence $\mu=|\lambda|(2k+n)+\lambda^2$ and

$$\lambda_k(\mu)=\frac{1}{2}\sqrt{4\mu+(2k+n)^2}-\frac{2k+n}{2}. \tag{2.12}$$

Therefore,

$$\mathcal{P}^{\Delta_{\mathbb{H}}}_\mu f(z,t)=\frac{1}{4^n\pi^{n+1}}\sum_{k=0}^\infty\frac{\left(\sqrt{4\mu+(2k+n)^2}-2k-n\right)^n}{\sqrt{4\mu+(2k+n)^2}}$$
$$\times\left(e^{-i\lambda_k(\mu)t}\,\Lambda_k^{\lambda_k(\mu)}f^{(\lambda_k(\mu))}(z)+e^{i\lambda_k(\mu)t}\,\Lambda_k^{-\lambda_k(\mu)}f^{(-\lambda_k(\mu))}(z)\right).$$

The operators $\Lambda_k^\lambda$, defined by (2.6), are the spectral projection of the twisted Laplacian on $\mathbb{C}^n$, that we now introduce. Given a Schwartz function $f$, consider the Fourier transform in the central variable of the functions $X_jf$ and $Y_jf$. Integrating by parts we obtain

$$(X_jf)^{(\lambda)}(z)=\left(\frac{\partial}{\partial x_j}+\frac{i}{2}\lambda y_j\right)f^{(\lambda)}(z),$$

and similarly for $Y_j$. Hence, setting

$$X_j^{(\lambda)}=\frac{\partial}{\partial x_j}+\frac{i}{2}\lambda y_j\quad\text{and}\quad Y_j^{(\lambda)}=\frac{\partial}{\partial x_j}-\frac{i}{2}\lambda y_j,$$

we have

$$(X_jf)^{(\lambda)}=X_j^{(\lambda)}f^{(\lambda)}\quad\text{and}\quad(Y_jf)^{(\lambda)}=Y_j^{(\lambda)}f^{(\lambda)}.$$

These formulae imply

$$(Lf)^{(\lambda)}=-\sum_{j=1}^n\left((X_j^{(\lambda)})^2+(Y_j^{(\lambda)})^2\right)f^{(\lambda)}=\Delta^{(\lambda)}f^{(\lambda)},$$

where $\Delta^{(\lambda)}$, for $\lambda\neq0$, is the $\lambda$-*twisted Laplacian*. Note that $\Delta^{(0)}$ is the Laplacian on $\mathbb{C}^n$.

From properties of the twisted convolution and of the Laguerre functions (see [18]), it follows that for any Schwartz function $g$ on $\mathbb{C}^n$

$$\Delta^{(\lambda)}\left(\Lambda_k^\lambda g\right)=|\lambda|(2k+n)\Lambda_k^\lambda g$$

and

$$g=\sum_{k=0}^\infty\Lambda_k^\lambda g.$$

Therefore, for $\lambda \neq 0$ the operators $\Lambda_k^\lambda$ are the spectral projections associated to $\Delta^{(\lambda)}$. When $\lambda = 1$, we will write for simplicity $\Lambda_k$, instead of $\Lambda_k^1$.

We now prove estimate (1.5) in Theorem 1.3 in the special case of the Heisenberg group, thus improving on (1.3) in Theorem 1.2. The first step is a simple lemma that easily follows from dilation arguments.

**Lemma 2.2.** *Suppose that $\Lambda_k : L^p(\mathbb{C}^n) \to L^q(\mathbb{C}^n)$ for some $p, q$. If $g$ lies in $\mathcal{S}(\mathbb{C}^n)$, then for all $\lambda > 0$ we have*

$$\left\|\Lambda_k^\lambda g\right\|_{L^q(\mathbb{C}^n)} \leq \lambda^{n\left(\frac{1}{p}-\frac{1}{q}-1\right)} \left\|\Lambda_k\right\|_{L^p(\mathbb{C}^n)\to L^q(\mathbb{C}^n)} \|g\|_{L^p(\mathbb{C}^n)}.$$

**Remark 2.3.** Observe that $\left\|\Lambda_k^{-1}\right\|_{L^p(\mathbb{C}^n)\to L^q(\mathbb{C}^n)} = \left\|\Lambda_k^1\right\|_{L^p(\mathbb{C}^n)\to L^q(\mathbb{C}^n)} = \left\|\Lambda_k\right\|_{L^p(\mathbb{C}^n)\to L^q(\mathbb{C}^n)}$.

Then we prove the following conditional statement.

**Proposition 2.4.** *Let $\mu > 0$. If $\Lambda_k$ is bounded from $L^p(\mathbb{C}^n)$ to $L^q(\mathbb{C}^n)$, then*

$$\left\|\mathcal{P}_\mu^m f\right\|_{L_t^\infty L_z^q} \leq \frac{2\|f\|_{L_t^1 L_z^p}}{(2\pi)^{n+1}} \left(\sum_{k=0}^\infty (\lambda_k(\mu))^{n\left(\frac{1}{p}-\frac{1}{q}\right)} \lambda_k'(\mu) \|\Lambda_k\|_{L^p(\mathbb{C}^n)\to L^q(\mathbb{C}^n)}\right) \tag{2.13}$$

*for all Schwartz functions $f$ on $\mathbb{H}_n$.*

**Proof.** To simplify the notations we write $f$ as if it were the product of two functions, that is $f(z, t) = h(t)g(z)$. Then (2.10) becomes

$$\mathcal{P}_\mu^m f(z,t) = \sum_{k=0}^\infty \frac{(\lambda_k(\mu))^n \lambda_k'(\mu)}{(2\pi)^{n+1}} \Big(e^{-i(\lambda_k(\mu))t}\, \hat{h}((\lambda_k(\mu)))\Lambda_k^{(\lambda_k(\mu))} g(z) + e^{i(\lambda_k(\mu))t}\, \hat{h}(-(\lambda_k(\mu)))\Lambda_k^{-(\lambda_k(\mu))} g(z)\Big).$$

Since $|\hat{h}(\lambda)| \leq \|h\|_{L^1(\mathbb{R})}$ for all $\lambda$, we obtain

$$\left|\mathcal{P}_\mu^m f(z,t)\right| \leq \|h\|_{L_t^1} \sum_{k=0}^\infty \frac{(\lambda_k(\mu))^n \lambda_k'(\mu)}{(2\pi)^{n+1}} \Big(|\Lambda_k^{(\lambda_k(\mu))} g(z)| + |\Lambda_k^{-\lambda_k(\mu)} g(z)|\Big).$$

Therefore, the triangle inequality implies

$$\left(\int_{\mathbb{C}^n} |\mathcal{P}_\mu^m f(z,t)|^q dz\right)^{\frac{1}{q}} \leq \frac{\|h\|_{L^1(\mathbb{R})}}{(2\pi)^{n+1}} \sum_{k=0}^\infty \lambda_k(\mu)^n \lambda_k'(\mu) \times \Big(\|\Lambda_k^{\lambda_k(\mu)} g\|_{L^q(\mathbb{C}^n)} + \|\Lambda_k^{-\lambda_k(\mu)} g\|_{L^q(\mathbb{C}^n)}\Big),$$

which by Lemma 2.2 and the subsequent observation, yields,

$$\left(\int_{\mathbb{C}^n} |\mathcal{P}_\mu^m f(z,t)|^q dz\right)^{\frac{1}{q}} \leq \frac{2}{(2\pi)^{n+1}} \|h\|_{L^1(\mathbb{R})} \|g\|_{L^p(\mathbb{C}^n)} \times \left(\sum_{k=0}^\infty (\lambda_k(\mu))^{n\left(\frac{1}{p}-\frac{1}{q}\right)} \lambda_k'(\mu) \|\Lambda_k^1\|_{L^p(\mathbb{C}^n)\to L^q(\mathbb{C}^n)}\right),$$

proving the statement. □

In order to control the convergence of the series in (2.13), we apply the sharp estimates for the $L^p - L^2$ norms, $1 \le p \le 2$, of the operators $\Lambda_k$ recently proved by H. Koch and F. Ricci, stating that

$$\|\Lambda_k\|_{L^p(\mathbb{C}^n)\to L^2(\mathbb{C}^n)} \lesssim C(2k+n)^{\gamma\left(\frac{1}{p}\right)}, \quad 1 \le p \le 2, \tag{2.14}$$

where $\gamma$ is the piecewise affine function on $[\frac{1}{2}, 1]$ defined by

$$\gamma\left(\frac{1}{p}\right) := \begin{cases} n\left(\frac{1}{p} - \frac{1}{2}\right) - \frac{1}{2} & \text{if } 1 \le p \le p_*, \\ \frac{1}{2}\left(\frac{1}{2} - \frac{1}{p}\right) & \text{if } p_* \le p \le 2, \end{cases}$$

with critical point $p_* = p_*(2n)$, defined by (1.1).

**Lemma 2.5.** *If* $1 \le p \le 2 \le q \le \infty$, *then*

$$\|\Lambda_k\|_{L^p(\mathbb{C}^n)\to L^q(\mathbb{C}^n)} \le C(2k+n)^{\gamma\left(\frac{1}{p}\right)+\gamma\left(\frac{1}{q'}\right)}. \tag{2.15}$$

**Proof.** By duality the estimates (2.14) are equivalent to

$$\|\Lambda_k\|_{L^2(\mathbb{C}^n)\to L^q(\mathbb{C}^n)} \le C(2k+n)^{\gamma\left(\frac{1}{q'}\right)}, \quad 2 \le q \le \infty,$$

yielding (2.15).

It easily follows from [5] that the above estimate is sharp for $1 \le p \le p_*(2n)$, $p'_*(2n) \le q \le \infty$ and for $p_*(2n) \le p \le 2$, $2 \le q \le p'_*(2n)$. □

In the case of the subLaplacian, from (2.13) it follows that

$$\left\|\mathcal{P}^L_\mu f\right\|_{L^\infty_t L^q_z} \le C\mu^{n\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\sum_{k=0}^{\infty}(2k+n)^{\gamma\left(\frac{1}{p}\right)+\gamma\left(\frac{1}{q'}\right)-n\left(\frac{1}{p}-\frac{1}{q}\right)-1}\right) \|f\|_{L^1_t L^p_z}, \tag{2.16}$$

for $1 \le p \le 2 \le q \le \infty$. To study the convergence of this series, we need to distinguish four cases according to the relative position of $p$ and $q$ with respect to the critical exponents $p_*(2n)$ and $p'_*(2n)$. We collect the result in the following lemma.

**Lemma 2.6.** *For any real number* $\alpha$ *we define*

$$\mathcal{S}_\alpha = \sum_{k=0}^{\infty}(2k+n)^{\gamma\left(\frac{1}{p}\right)+\gamma\left(\frac{1}{q'}\right)-n\left(\frac{1}{p}-\frac{1}{q}\right)+\alpha}.$$

*Then we have the following.*

(I) *For* $1 \le p < p_*$, $2 \le q \le p'_*$ *the series* $\mathcal{S}_\alpha$ *converges if* $\alpha < \frac{2n+1}{2}\left(\frac{1}{p'_*} - \frac{1}{q}\right)$.

(II) *For* $1 \le p < p_*$ *and* $p'_* \le q \le \infty$ *the series* $\mathcal{S}_\alpha$ *converges if* $\alpha < 0$.

(III) *For* $p_* \le p \le 2$ *and* $2 \le q \le p'_*$ *the series* $\mathcal{S}_\alpha$ *converges if* $\alpha < \frac{2n+1}{2}\left(\frac{1}{p} - \frac{1}{q}\right) - 1$.

(IV) *For* $p_* \le p \le 2$ *and* $p'_* \le q \le \infty$ *the series* $\mathcal{S}_\alpha$ *converges if* $\alpha < \frac{2n+1}{2}\left(\frac{1}{p} - \frac{1}{p_*}\right)$.

**Proof.** In order not to burden the exposition, we prove only (I), the other cases being analogous. The series converges if

$$\gamma\left(\frac{1}{p}\right)+\gamma\left(\frac{1}{q'}\right)-n\left(\frac{1}{p}-\frac{1}{q}\right)<-1-\alpha. \tag{2.17}$$

If $1\leq p<p_*$, $2\leq q\leq p_*'$ we have

$$\gamma\left(\frac{1}{p}\right)+\gamma\left(\frac{1}{q'}\right)-n\left(\frac{1}{p}-\frac{1}{q}\right)=\frac{2n+1}{2q}-\frac{2n+3}{4}.$$

Thus, the condition (2.17) becomes

$$\alpha<\frac{2n+1}{2}\left(\frac{2n-1}{2(2n+1)}-\frac{1}{q}\right)=\frac{2n+1}{2}\left(\frac{1}{p_*'}-\frac{1}{q}\right)$$

proving (I). □

Estimate (2.16) and Lemma 2.6 with $\alpha=-1$ entail the following result.

**Theorem 2.7.** *For all pairs $(p,q)$ with $1\leq p\leq 2$, $2\leq q\leq\infty$, and $(p,q)\neq(2,2)$, there is a constant $C_{pq}$, such that for all Schwartz functions $f$ and all positive numbers $\mu$ we have the inequality*

$$\left\|\mathcal{P}_\mu^L f\right\|_{L_t^\infty L_z^q}\leq C_{pq}\mu^{n\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_{L_t^1L_z^p}.$$

**Remark 2.8.** Note that in the four cases listed above we can insert a positive power of $k$ in the series still preserving the convergence. This amounts to study a derivative $\partial_t^{-\alpha}$ of negative order of $\mathcal{P}_\mu^L f$ and estimate its norm. In other words, instead of considering the operator $\mathcal{P}_\mu^L$ given by (2.11), we introduce the operator

$$\partial_t^{-\alpha}\mathcal{P}_\mu^L f(z,t)=\frac{\mu^{n-\alpha}}{(2\pi)^{n+1}}\sum_{k=0}^{\infty}(n+2k)^{\alpha-(n+1)}\times\left(e^{-i\lambda_k(\mu)t}\Lambda_k^{\lambda_k(\mu)}f^{(\lambda_k(\mu))}(z)+e^{i\lambda_k(\mu)t}\Lambda_k^{-\lambda_k(\mu)}f^{(-\lambda_k(\mu))}(z)\right)$$

and prove estimates like those in the theorem for $\alpha$ small. It is easy to see that this yields the spectral resolution of the operator

$$\left(\frac{n+1-\alpha}{n+1}\right)^{\frac{1}{n+1-\alpha}}|T|^{\frac{\alpha}{n+1-\alpha}}L.$$

Then, retracing the argument that lead to (2.16), we obtain

$$\left\|\partial_t^{-\alpha}\mathcal{P}_\mu^L f\right\|_{L_t^\infty L_z^q}\leq C\mu^{n\left(\frac{1}{p}-\frac{1}{q}\right)-\alpha}\left(\sum_{k=0}^{\infty}(2k+n)^{\alpha+\gamma\left(\frac{1}{p}\right)+\gamma\left(\frac{1}{q'}\right)-n\left(\frac{1}{p}-\frac{1}{q}\right)-1}\right)\|f\|_{L_t^1L_z^p}.$$

From this estimate, using Lemma 2.6, we obtain the following theorem. We omit the argument which is similar to that of the previous theorem.

**Theorem 2.9.** *The estimate*

$$\left\| \partial_t^{-\alpha} \mathcal{P}_\mu^L f \right\|_{L_t^\infty L_z^q} \le C \mu^{n\left(\frac{1}{p}-\frac{1}{q}\right)-\alpha} \|f\|_{L_t^1 L_z^p} \tag{2.18}$$

*holds:*

(I) *When* $\alpha < \frac{2n+1}{2}\left(\frac{1}{p'_*}-\frac{1}{q}\right)+1$ *if* $1 \le p \le p_*$ *and* $2 \le q \le p'_*$.
(II) *When* $\alpha < \frac{2n+1}{2}\left(\frac{1}{p}-\frac{1}{p_*}\right)+1$ *if* $p_* \le p \le 2$ *and* $p'_* \le q \le \infty$.
(III) *When* $\alpha < 1$ *if* $1 \le p \le p_*$ *and* $p'_* \le q \le \infty$.
(IV) *When* $\alpha < \frac{2n+1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)$ *if* $p_* \le p \le 2$ *and* $2 \le q \le p'_*$.

We remark that according to the above theorem $\alpha$ need always to be strictly smaller than 1. This is consistent with Müller's counterexample showing that in the estimate (1.3) one cannot find nothing better than the $L^\infty$-norm in the central variable. Indeed, if we had (2.18) with $\alpha = 1$, then the $t$-antiderivative of $\mathcal{P}_\mu^L f$ would be bounded.

## 3. Restriction estimates for the full Laplacian on the Heisenberg group

Similar to what we have done so far for the subLaplacian, we now study the case of the full Laplacian on $\mathbb{H}_n$. We consider here only the estimates for $q = 2$. A more detailed discussion of restriction estimates for the full Laplacian in the more general framework of Métivier groups may be found in [2].

**Theorem 3.1.** *For* $1 \le p \le p_*$ *we have*

$$\left\| \mathcal{P}_\mu^{\Delta_{\mathbb{H}}} f \right\|_{L_t^\infty L_z^2} \le C \mu^{n\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{1}{4}} \|f\|_{L_t^1 L_z^p} \tag{3.1}$$

*and for* $p_* \le p \le 2$ *we have*

$$\left\| \mathcal{P}_\mu^{\Delta_{\mathbb{H}}} f \right\|_{L_t^\infty L_z^2} \le C \mu^{\frac{2n-1}{4}\left(\frac{1}{p}-\frac{1}{2}\right)} \|f\|_{L_t^1 L_z^p}. \tag{3.2}$$

**Proof.** Plugging (2.12) in (2.13) we obtain

$$\begin{aligned}
\left\| \mathcal{P}_\mu^{\Delta_{\mathbb{H}}} f \right\|_{L_t^\infty L_z^2} &\le C \|f\|_{L_t^1 L_z^p} \\
&\quad \times \left( \sum_{k=0}^\infty \|\Lambda_k\|_{L^p(\mathbb{C}^n)\to L^2(\mathbb{C}^n)} \frac{\left(\sqrt{4\mu+(2k+n)^2}-2k-n\right)^{n\left(\frac{1}{p}-\frac{1}{2}\right)}}{\sqrt{4\mu+(2k+n)^2}} \right) \\
&\le C \mu^{n\left(\frac{1}{p}-\frac{1}{2}\right)} \|f\|_{L_t^1 L_z^p} \left( \sum_{k=0}^\infty \frac{\|\Lambda_k\|_{L^p(\mathbb{C}^n)\to L^2(\mathbb{C}^n)}}{\sqrt{4\mu+(2k+n)^2}} \right. \\
&\quad \left. \times \left(\sqrt{4\mu+(2k+n)^2}+2k+n\right)^{-n\left(\frac{1}{p}-\frac{1}{2}\right)} \right).
\end{aligned}$$

We split the sum into the sum over those $k$ such that $2k+n \le 2\sqrt{\mu}$ and those such that $2k+n > 2\sqrt{\mu}$. Then we control the first term, say $I$, by

$$\begin{aligned} I &\le C\mu^{n\left(\frac{1}{p}-\frac{1}{2}\right)} \|f\|_{L^1_t L^p_z} \mu^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{1}{2}} \left( \sum_{2k+n\le 2\sqrt{\mu}} \|\Lambda_k\|_{L^p(\mathbb{C}^n)\to L^2(\mathbb{C}^n)} \right) \\ &\le C\mu^{\frac{n}{2}\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{1}{2}} \|f\|_{L^1_t L^p_z} \left( \sum_{2k+n\le 2\sqrt{\mu}} \|\Lambda_k\|_{L^p(\mathbb{C}^n)\to L^2(\mathbb{C}^n)} \right) \end{aligned}$$

and the second, say $II$, by

$$II \le C\mu^{n\left(\frac{1}{p}-\frac{1}{2}\right)} \|f\|_{L^1_t L^p_z} \left( \sum_{2k+n\ge 2\sqrt{\mu}} \frac{\|\Lambda_k\|_{L^p(\mathbb{C}^n)\to L^2(\mathbb{C}^n)}}{(2k+n)^{1+n\left(\frac{1}{p}-\frac{1}{2}\right)}} \right).$$

When $1 \le p \le p_*$ by (2.14) we have

$$\begin{aligned} I &\le C\mu^{\frac{n}{2}\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{1}{2}} \|f\|_{L^1_t L^p_z} \left( \sum_{2k+n\le 2\sqrt{\mu}} (2k+n)^{n\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{1}{2}} \right) \\ &\le C\mu^{n\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{1}{4}} \|f\|_{L^1_t L^p_z} \end{aligned}$$

and

$$\begin{aligned} II &\le C\mu^{n\left(\frac{1}{p}-\frac{1}{2}\right)} \|f\|_{L^1_t L^p_z} \left( \sum_{2k+n\ge 2\sqrt{\mu}} (2k+n)^{-\frac{3}{2}} \right) \\ &\le C\mu^{n\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{1}{4}} \|f\|_{L^1_t L^p_z} \end{aligned}$$

proving (3.1).

When $p_* \le p \le 2$ we have

$$\begin{aligned} I &\le C\mu^{\frac{n}{2}\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{1}{2}} \|f\|_{L^1_t L^p_z} \left( \sum_{2k+n\le 2\sqrt{\mu}} (2k+n)^{\frac{1}{2}\left(\frac{1}{2}-\frac{1}{p}\right)} \right) \\ &\le C\mu^{\frac{(2n-1)}{4}\left(\frac{1}{p}-\frac{1}{2}\right)} \|f\|_{L^1_t L^p_z} \end{aligned}$$

and

$$\begin{aligned} II &\le C\mu^{n\left(\frac{1}{p}-\frac{1}{2}\right)} \|f\|_{L^1_t L^p_z} \left( \sum_{2k+n\ge 2\sqrt{\mu}} (2k+n)^{-1-\frac{2n+1}{2}\left(\frac{1}{p}-\frac{1}{2}\right)} \right) \\ &\le C\mu^{\frac{2n-1}{4}\left(\frac{1}{p}-\frac{1}{2}\right)} \|f\|_{L^1_t L^p_z} \end{aligned}$$

proving (3.2). $\square$

## 4. Spectral resolution of the subLaplacian on Métivier groups

Let $G$ be a connected, simply connected, two-step nilpotent Lie group, with Lie algebra $\mathfrak{g}$. We denote the centre of $\mathfrak{g}$ by $\mathfrak{z}$ and set $\dim \mathfrak{z} = d$. If $\omega \in \mathfrak{z}^*$, the dual of $\mathfrak{z}$, we define

$$\mathfrak{g}_\omega = \mathfrak{g}/\ker \omega.$$

Since $\ker \omega$, being a subspace of the centre, is an ideal in $\mathfrak{g}$, $\mathfrak{g}_\omega$ is a Lie algebra. The connected simply connected subgroup of $G$ with Lie algebra $\mathfrak{g}_\omega$ will be denoted by $G_\omega$.

Let $\mathfrak{v}$ be a complement of $\mathfrak{z}$ in $\mathfrak{g}$. We assume that $G$ satisfies a non-degeneracy condition, which is expressed in terms of the bilinear application $B_\omega(X, Y) = \omega([X, Y])$, with $X, Y$ in $\mathfrak{v}$ and $\omega$ in $S$. Recall that $B_\omega$ is *non-degenerate* if the space $\{V \in \mathfrak{v} : \omega([V, U]) = 0 \text{ for all } U \text{ in } \mathfrak{v}\}$ is trivial.

**Definition 4.1** *([6]).* We say that $G$ is a Métivier group if $B_\omega$ is non-degenerate for all $\omega \neq 0$.

In this case the dimension of $\mathfrak{v}$ is even, say $\dim \mathfrak{v} = 2n$, and $G_\omega$ is isomorphic to the Heisenberg group $\mathbb{H}_n$ with Lie algebra $\mathfrak{h}_n = \mathbb{R} \oplus \mathfrak{v}_n$, $\mathfrak{v}_n = \mathbb{R}^{2n}$. Moreover, $\mathfrak{v}$ generates $\mathfrak{g}$ as a Lie algebra.

Only for notational convenience we introduce an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$, with the property that $\mathfrak{z}$ and $\mathfrak{v}$ are orthogonal subspaces. Let $|\cdot|$ denote the norm induced by $\langle \cdot, \cdot \rangle$ on $\mathfrak{z}^*$, the dual of $\mathfrak{z}$. We call $S$ the unit sphere in $\mathfrak{z}^*$, that is,

$$S := \{\omega \in \mathfrak{z}^* \ : \ |\omega| = 1\}.$$

For any fixed $\omega$ in $S$ there is an element $Z_\omega$ in $\mathfrak{z}$ such that $\omega(Z_\omega) = 1$ and $|Z_\omega| = 1$. Indeed, by definition

$$|\omega| = \sup_{|Z|=1} |\omega(Z)|$$

and, since $\omega$ is continuous, there exists $Z_\omega \in \mathfrak{z}$ such that $|Z_\omega| = 1$ and $\omega(Z_\omega) = 1$.

The centre of the Lie algebra decomposes into the sum

$$\mathfrak{z} = \mathbb{R} Z_\omega \oplus \ker \omega. \tag{4.1}$$

Observe that for every $Z \in \ker \omega$ we have $\langle Z_\omega, Z \rangle = 0$. We shall systematically identify the quotient $\mathfrak{z}/\ker \omega$ with $\mathbb{R} Z_\omega$. Then $\mathbb{R} Z_\omega \oplus \mathfrak{v}$ is a Lie algebra isomorphic to $\mathfrak{g}_\omega$.

Since $\mathfrak{g}$ is nilpotent, the exponential map, $\exp : \mathfrak{g} \to G$, is surjective. Thus we may parametrize $G$ by $\mathfrak{v} \oplus \mathfrak{z}$, endowing it with the exponential coordinates. More precisely, we fix a basis $\{Z_1, \ldots, Z_d, V_1, \ldots, V_{2n}\}$ of $\mathfrak{g}$, with $\{Z_1, \ldots, Z_d\}$ a basis of $\mathfrak{z}$ and $\{V_1, \ldots, V_{2n}\}$ a basis of $\mathfrak{v}$, and identify a point $g$ of $G$ with the point $(V, Z)$ in $\mathbb{R}^k \times \mathbb{R}^d$, such that

$$g = \exp(V, Z) = \exp\left( \sum_{j=1}^{2n} v_j V_j + \sum_{a=1}^{d} z_a Z_a \right).$$

In these coordinates the product law is given by the Baker–Campbell–Hausdorff formula

$$(V, Z)(V', Z') = \left( V + V', Z + Z' + \frac{1}{2}[V, V'] \right),$$

for all $V, V' \in \mathfrak{v}$ and $Z, Z' \in \mathfrak{z}$.

If we denote by $dV$ and $dZ$ the Lebesgue measures on $\mathfrak{v}$ and $\mathfrak{z}$ respectively, then the product measure $dV\,dZ$ is a left-invariant Haar measure on $G$. We shall denote by $L^p(G)$ the corresponding Lebesgue spaces.

Finally, we call $\mathcal{S}(G)$ the Schwartz space on $G$, that is, the space of functions $f$ on $G$ such that $f \circ \exp$ belongs to the usual Schwartz space on the Euclidean space $\mathfrak{g}$.

To any vector in $\mathfrak{g}$, thought of as the tangent space to $G$ at the origin, we associate a left-invariant vector field on $G$. If $f \in \mathcal{S}(G)$, $V = \sum_{j=1}^{2n} v_j V_j$, $T = \sum_{a=1}^{d} z_a Z_a$, we set

$$\begin{aligned} \tilde{V}_j f(V,T) &= \frac{d}{ds} f\Big((V,T)(sV_j,0)\Big)\Big|_{s=0} \\ &= \frac{\partial f}{\partial v_j}(V,T) + \frac{1}{2}\sum_{a=1}^{d} \langle Z_a, [V,V_j]\rangle \frac{\partial f}{\partial z_a}(V,T), \end{aligned}$$

and

$$\begin{aligned} \tilde{T}_a f(V,T) &= \frac{d}{ds} f\Big((V,T)(0,sT_a)\Big)\Big|_{s=0} \\ &= \frac{\partial f}{\partial z_a}(V,T). \end{aligned}$$

Then the vectors fields

$$\tilde{V}_j = \frac{\partial}{\partial v_j} + \frac{1}{2}\sum_{a=1}^{d} \langle Z_a, [V,V_j]\rangle \frac{\partial}{\partial z_a}, \qquad \tilde{T}_a = \frac{\partial}{\partial z_a}$$

are left invariant.

In terms of these vectors we define the subLaplacian

$$L = -\tilde{V}_1^2 - \cdots - \tilde{V}_{2n}^2,$$

the Laplacian on the centre

$$\Delta_{\mathfrak{z}} = -\tilde{T}_1^2 - \cdots - \tilde{T}_d^2,$$

and the full Laplacian

$$\Delta_G = L + \Delta_{\mathfrak{z}}.$$

The operators $L$ and $\Delta_G$ are positive and essentially self-adjoint on $L^2(G)$. Moreover, since the set of vector fields $\{\tilde{V}_1, \tilde{V}_2, \ldots, \tilde{V}_{2n}\}$ generates $\mathfrak{g}$ as a Lie algebra, $L$ and $\Delta_G$ are hypoelliptic.

We will obtain the spectral decompositions of $L$ and $\Delta_G$ from those of the subLaplacian $L_{\mathbb{H}}$ and of the full Laplacian $\Delta_{\mathbb{H}}$ on the Heisenberg group $\mathbb{H}_n$, by means of a partial Radon transform in the central variables.

**Definition 4.2.** For any function $f$ in $\mathcal{S}(G)$ and for $\omega \in S$, we set

$$R_\omega f(V,t) := \int_{\{Z' \in \ker\omega\}} f(V, tZ_\omega + Z')\, dZ',$$

where $dZ'$ denotes the Lebesgue measure on the hyperplane $\ker\omega$ in $\mathfrak{z}$.

For each $\omega$ in $S$, $R_\omega f$ is a function on the subgroup $G_\omega$, which is isomorphic to $\mathbb{H}_n$. According to the Euclidean theory, the family of functions $\{R_\omega f\}_{\omega\in S}$ completely determines $f$.

Fix $\omega$ in $S$. If we choose the basis of $\mathfrak{z}$ in such a way that $Z_1 = Z_\omega$, we have

$$\int_{\ker\omega} \frac{\partial f}{\partial z_a}(V, tZ_\omega + Z')dZ' = 0, \tag{4.2}$$

for all $a = 2, \dots, d$, since $f(V, Z)$ vanishes as the norm $|Z|$ goes to infinity. Moreover,

$$\int_{\ker\omega} \frac{\partial f}{\partial t}(V, tZ_\omega + Z')dZ' = \frac{\partial}{\partial t}\int_{\ker\omega} f(V, tZ_\omega + Z')dZ' = \frac{\partial}{\partial t} R_\omega f(V, t).$$

Hence, setting $T^{(\omega)} = \frac{\partial}{\partial t}$, we have

$$R_\omega\big(\tilde{Z}_1 f\big)(V, t) = \frac{\partial}{\partial t} R_\omega f(V, t) = T^{(\omega)} R_\omega f(V, t)$$

and

$$R_\omega\big(\tilde{Z}_a f\big)(V, t) = 0,$$

for all $a = 2, \dots, d$.

We have also

$$\begin{aligned} R_\omega\left(\tilde{V}_j f\right)(V, t) &= \int_{\ker\omega} \big(\tilde{V}_j f\big)(V, tZ_1 + Z')dZ' \\ &= \left(\frac{\partial}{\partial v_j} R_\omega f\right)(V, t) + \frac{1}{2}\langle Z_\omega, [V, V_j]\rangle \left(\frac{\partial}{\partial z_1} R_\omega f\right)(V, t) \\ &\quad + \frac{1}{2}\sum_{a=2}^{d}\langle Z_a, [V, V_j]\rangle \int_{\ker\omega} \frac{\partial f}{\partial z_a}(V, tZ_\omega + Z')dZ'. \end{aligned}$$

Since $\omega(Z) = \langle Z_\omega, Z\rangle$ for $Z \in \mathfrak{z}$, we have $\langle Z_\omega, [V, V_j]\rangle = \omega([V, V_j])$. Then it follows from (4.2) that

$$R_\omega\left(\tilde{V}_j f\right)(V, t) = \left(\frac{\partial}{\partial v_j} + \frac{1}{2}\omega\big([V, V_j]\big)\frac{\partial}{\partial t}\right)\big(R_\omega f\big)(V, t). \tag{4.3}$$

Setting

$$V_j^{(\omega)} := \frac{\partial}{\partial v_j} + \frac{1}{2}\omega\big([V, V_j]\big)\frac{\partial}{\partial t}, \quad j = 1, \dots, k,$$

(4.3) is equivalent to the following commutation relation

$$R_\omega\left(\tilde{V}_j f\right) = V_j^{(\omega)}\left(R_\omega f\right), \tag{4.4}$$

for all $f$ in $\mathcal{S}(G)$. In other words the vector fields $V_j^{(\omega)}$ are the projections on the group $G_\omega$ of the vectors $\tilde{V}_j$.

The vector fields $\{V_1^{(\omega)}, \dots, V_{2n}^{(\omega)}\}$ together with $T^{(\omega)} = \frac{\partial}{\partial t}$ yield a basis of left-invariant fields on the group $G_\omega$. In terms of them we introduce the subLaplacian

$$L^{(\omega)} = -\big(V_1^{(\omega)}\big)^2 - \cdots - \big(V_k^{(\omega)}\big)^2, \tag{4.5}$$

on $G_\omega$. We then have

$$R_\omega(Lf) = L^{(\omega)}(R_\omega f), \tag{4.6}$$

for all functions $f \in \mathcal{S}(G)$. From this relation we may easily obtain the spectral decomposition of $f$ from that of $R_\omega f$ by means of the inverse of the Radon transform.

To carry out this plan, we introduce the partial Fourier transform in the central variable of a function $g$ in $\mathcal{S}(G_\omega)$,

$$\mathfrak{F}_1 g(V;\lambda) = \int_{-\infty}^{\infty} e^{i\lambda t} g(V,t)dt, \tag{4.7}$$

$\lambda \in \mathbb{R}$. An integration by parts then yields

$$\begin{aligned}\mathfrak{F}_1\Big({V_j}^{(\omega)} g\Big)(V;\lambda) &= \int_{-\infty}^{\infty} e^{i\lambda t} V_j^{(\omega)} g(V,t)dt \\ &= \left(\frac{\partial}{\partial v_j} - \frac{i}{2}\lambda\omega\big([V,V_j]\big)\right)\mathfrak{F}_1 g(V;\lambda).\end{aligned}$$

Defining

$$V_j^{(\lambda,\omega)} := \frac{\partial}{\partial v_j} - \frac{i}{2}\lambda\omega\big([V,V_j]\big), \quad j = 1,\ldots,k, \tag{4.8}$$

we have proved that

$$\mathfrak{F}_1\left({V_j}^{(\omega)} g\right) = V_j^{(\lambda,\omega)}\mathfrak{F}_1\big(g\big), \quad j = 1,\ldots,k, \tag{4.9}$$

which implies

$$\mathfrak{F}_1\Big(L^{(\omega)} g\Big)(V;\lambda) = \varDelta^{(\lambda,\omega)}\mathfrak{F}_1 g(V;\lambda),$$

where

$$\varDelta^{(\lambda,\omega)} = -(V_1^{(\lambda,\omega)})^2 \cdots - (V_k^{(\lambda,\omega)})^2 \tag{4.10}$$

is the $\lambda$-*twisted Laplacian* associated to the group $G_\omega$.

In the Euclidean setting it is well known that the Radon transform factorizes the Fourier transform. Correspondingly on the group $G$ we have for $\omega \in S$ and $\lambda \in \mathbb{R}$

$$\begin{aligned}\mathfrak{F}_1\big(R_\omega f\big)(V;\lambda) &= \int_{-\infty}^{\infty}\int_{\ker\omega} e^{i\lambda\,\omega(tZ_\omega + Z')} f(V, tZ_\omega + Z')dZ'dt \\ &= \int_{\mathfrak{z}} e^{i\lambda\,\omega(Z)} f(V,Z)dZ = \mathfrak{F}_{\mathfrak{z}} f(V,\lambda\omega),\end{aligned} \tag{4.11}$$

where $\mathfrak{F}_{\mathfrak{z}} f$ denotes the Fourier transform of $f$ along the central variables.

Similarly, setting $\eta_a = \eta(Z_a)$ for $\eta \in \mathfrak{z}^*$, we obtain

$$\begin{aligned}\mathfrak{F}_{\mathfrak{z}}(\tilde{V}_j f)(V,\eta) &= \int_{\mathfrak{z}} e^{i\eta(Z)}\tilde{V}_j f(V,Z)dZ \\ &= \frac{\partial}{\partial v_j}\mathfrak{F}_{\mathfrak{z}} f(V,\eta) + \frac{1}{2}\sum_{a=1}^{d}\langle Z_a, [V_j, V]\rangle \int_{\mathfrak{z}} e^{i\eta(Z)}\frac{\partial f}{\partial z_a}(V,Z)dZ \\ &= \frac{\partial}{\partial v_j}\mathfrak{F}_{\mathfrak{z}} f(V,\eta) - \frac{i}{2}\sum_{a=1}^{d}\eta_a\langle Z_a, [V,V_j]\rangle(\mathfrak{F}_{\mathfrak{z}} f)(V,\eta)\end{aligned}$$

$$= \frac{\partial}{\partial v_j}\mathfrak{F}_\mathfrak{z} f(V,\eta) - \frac{i}{2}\eta\left([V,V_j]\right)(\mathfrak{F}_\mathfrak{z} f)(V,\eta)$$
$$= V_j^\eta(\mathfrak{F}_\mathfrak{z} f)(V,\eta).$$

Defining

$$V_j^\eta = \frac{\partial}{\partial v_j} - \frac{i}{2}\eta([V,V_j]), \quad j=1,\ldots,k, \tag{4.12}$$

we have showed that

$$\mathfrak{F}_\mathfrak{z}(\tilde{V}_j f) = V_j^\eta(\mathfrak{F}_\mathfrak{z} f).$$

Writing in polar coordinates $\eta$ as $\eta = \rho\omega$, with $\rho \ge 0$ and $\omega$ in $S$, we clearly have

$$V_j^\eta = V_j^{\rho\omega} = V_j^{\rho,\omega}, \tag{4.13}$$

where the vector fields $V_j^{\rho,\omega}$ have been defined in (4.8).

**Definition 4.3.** Given $\eta \in \mathfrak{z}^\star$, the $\eta$-twisted Laplacian of $G$ is

$$\Delta_\eta = -\left(V_1^\eta\right)^2 - \cdots - \left(V_k^\eta\right)^2. \tag{4.14}$$

**Lemma 4.4.** *Take $\eta = \rho\omega$ with $\omega \in S$ and $\rho > 0$. If $L^{(\omega)}$ is the subLaplacian defined by (4.5) on $G_\omega$, we have*

$$\Delta_{\rho\omega} = \Delta^{(\rho,\omega)} = (L^{(\omega)})^\rho, \tag{4.15}$$

*that is, the $(\rho\omega)$-twisted Laplacian of $G$, defined by (4.14), coincides with the $\rho$-twisted Laplacian, $\Delta^{(\rho,\omega)}$ of the group $G_\omega$, defined by (4.10).*

**Proof.** The first identity simply follows from (4.13) and the definitions (4.10) and (4.14).

For the second, using (4.11), (4.4) and (4.9), which here give

$$\mathfrak{F}_1\left(R_\omega \tilde{V}_j f\right)(V;\rho) = \mathfrak{F}_1(V_j^{(\omega)} R_\omega f)(V;\rho) = V_j^{\rho\omega}\Big(\mathfrak{F}_1\left(R_\omega f\right)\Big)(V;\rho)$$

we obtain

$$V_j^{\rho\omega} = (R_\omega \tilde{V}_j)^\rho = \frac{\partial}{\partial v_j} - \frac{i}{2}\rho\,\omega(\langle [V,V_j]\rangle),$$

whence $\Delta^{(\rho,\omega)} = (L^{(\omega)})^\rho$ follows. □

We write, for $j = 1,\ldots,2n$,

$$V_j^{\rho\omega} = \frac{\partial}{\partial v_j} - i\rho\sum_{k=1}^{2n} v_k\,\omega\left([V_k,V_j]\right)$$
$$= \frac{\partial}{\partial v_j} - i\rho\sum_{k=1}^{2n} v_k B_{kj}^\omega,$$

where $B_{jk}^\omega = \omega\left([V_j,V_k]\right) = -B_{kj}^\omega$ are the entries of the matrix $B_\omega$, introduced in Definition 4.1.

Since $B_\omega$ is non-degenerate and skew-symmetric, there exists a $2n \times 2n$ invertible matrix, $A_\omega$, such that

$$B_\omega\,(A_\omega V_1, A_\omega V_2) = J(V_1, V_2),$$

for all $V_1, V_2$ in $\mathfrak{v}$, where $J(X, Y)$ is the standard symplectic form corresponding to the matrix

$$\mathbb{J}_{2n} = \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix},$$

where $\mathbb{I}_n$ denotes the $n$-dimensional identity matrix. It follows that

$$(\det A_\omega)^2 = \frac{1}{|\det B_\omega|}.$$

Since $\det B_\omega$, which is a polynomial function in the components of $\omega$, never vanishes on the unit sphere, there is a positive constant $K$ such that

$$\frac{1}{K} \le |\det B_\omega|^{-\frac{1}{2}} = |\det A_\omega| \le K, \tag{4.16}$$

for $\omega$ in $S$.

We introduce a new set of coordinates $y_j^\omega$, defined in terms of $A_\omega$ by

$$y_j^\omega := \sum_{k=1}^{2n} (A_\omega)_{jk}\, v_k, \quad \text{for } j = 1, \ldots, 2n,$$

where $(A_\omega)_{jk}$ denotes the $(j, k)$-entry of $A_\omega$.

Correspondingly we define the vector fields

$$Y_j^{\rho\omega} := \sum_{k=1}^{2n} (A_\omega)_{kj} V_k^{\rho\omega}, \quad j = 1, \ldots, 2n.$$

When only a point $\omega$ in $S$ is considered, we shall often suppress the index $\omega$ from $y_j^\omega$ and $Y_j^{\rho\omega}$, simply writing $y_j$ and $Y_j^\rho$.

One easily checks that

$$Y_j^\rho = \frac{\partial}{\partial y_j} + i\rho \sum_{k=1}^{2n} y_k\, \mathbb{J}_{jk}, \quad j = 1, \ldots, 2n,$$

or, more explicitly,

$$Y_k^\rho = \frac{\partial}{\partial y_k} + i\rho\, y_{k+n}, \qquad Y_{k+n}^\rho = \frac{\partial}{\partial y_{k+n}} - i\rho\, y_k, \quad k = 1, \ldots, n.$$

Therefore, the operator

$$\sum_{j=1}^{2n} \left(Y_j^\rho\right)^2 = -\sum_{j=1}^{2n} \frac{\partial^2}{\partial y_j^2} + i\rho \sum_{j=1}^{n} \left( y_j \frac{\partial}{\partial y_{j+n}} - y_{j+n} \frac{\partial}{\partial y_j} \right) + \rho^2 \sum_{j=1}^{2n} y_j^2$$

coincides with the $\rho$-twisted Laplacian $\Delta_{\mathbb{H}}^\rho$ associated to the Heisenberg group $\mathbb{H}_n$.

From now on we shall consider the coordinates $y_j$ as a fixed coordinate system on $\mathfrak{v}_n$, the orthogonal complement of the centre in $\mathfrak{h}_n$, interpreting $A_\omega$ as a family of linear invertible maps from $\mathfrak{v}$ to $\mathfrak{v}_n$ depending on $\omega$. Correspondingly, the family of vector fields $\{Y_j^\rho\}_{j=1}^{2n}$ will be thought of as a basis of vector fields on $\mathfrak{v}_n$.

**Proposition 4.5.** *Let $\omega \in S$ and $\rho > 0$. If $g$ is a function in $\mathcal{S}(\mathfrak{v})$, set $g_\omega = g \circ A_\omega^{-1}$.*

(a) *The spectral projection $\Lambda_k^{\rho\omega}$ onto the eigenspace of the $\rho\omega$-twisted Laplacian, $\Delta_{\rho\omega}$ (defined by (4.14)), corresponding to the eigenvalue $\rho(2k+n)$, is given by*

$$\Pi_k^{\rho\omega} g = \left(g_\omega \times_\rho \varphi_k^\rho\right) \circ A_\omega = \left(\Lambda_k^\rho g_\omega\right) \circ A_\omega, \tag{4.17}$$

*where $\Lambda_k^\rho$, defined by (2.6), is the spectral projection of the $\rho$-twisted Laplacian, $\Delta_{\mathbb{H}}^\rho$, on the Heisenberg group.*

(b) *Moreover,*

$$g(V) = \rho^n \sum_{k=0}^{\infty} \left(g_\omega \times_\rho \varphi_k^\rho\right)(A_\omega V) = \rho^n \sum_{k=0}^{\infty} \left(\Pi_k^{\rho\omega} g\right)(V). \tag{4.18}$$

**Proof.** Since

$$V_k^{\rho\omega} = \sum_{j=1}^{2n} (A_\omega)_{kj} Y_j^\rho,$$

we have

$$\Delta_{\rho\omega}\left(g \circ A_\omega\right) = \left(\Delta_{\mathbb{H}}^\rho g\right) \circ A_\omega.$$

Thus, writing $g = g_\omega \circ A_\omega$, we obtain

$$\Delta_{\rho\omega} g = \Delta_{\rho\omega}\left(g_\omega \circ A_\omega\right) = \left(\Delta_{\mathbb{H}}^\rho g_\omega\right) \circ A_\omega,$$

which implies

$$\begin{aligned} \Delta_{\rho\omega}\Big(\left(\Lambda_k^\rho g_\omega\right) \circ A_\omega\Big) &= \Big(\Delta_{\mathbb{H}}^\rho\left(\Lambda_k^\rho g_\omega\right)\Big) \circ A_\omega \\ &= \rho(2k+n)\left(\Lambda_k^\rho g_\omega\right) \circ A_\omega. \end{aligned}$$

Similarly, we obtain (4.18)

$$g(V) = \rho^n \sum_{k=0}^{\infty} \left(\Lambda_k^\rho g_\omega\right)(A_\omega V) = \rho^n \sum_{k=0}^{\infty} \left(g_\omega \times_\rho \varphi_k^\rho\right)(A_\omega V).$$

This is the expansion of $g$ in terms of the eigenfunctions of the $\rho\omega$-twisted Laplacian and the spectral projection onto the eigenspace associated to the eigenvalue $\rho(2k+n)$ of $\Delta^{\rho\omega}$ is thus given by (4.17). □

**Remark 4.6.** For each $\omega \in S$ the function $g_\omega = g \circ A_\omega^{-1}$ in (4.17) may be thought of as a function on the fixed reference space $\mathfrak{v}_n$.

We shall estimate the norms of the projections $\Pi_k^{\rho\omega}$ in terms of those of $\Lambda_k$ by means of the following lemma.

**Lemma 4.7.** *Fix $\omega$ in $S$. Suppose that $\Lambda_k : L^p(\mathfrak{v}_n) \to L^q(\mathfrak{v}_n)$ for some $p, q$. The following inequality holds*

$$\|\Pi_k^{\rho\omega} g\|_{L^q(\mathfrak{v})} \le C\rho^{n\left(\frac{1}{p}-\frac{1}{q}-1\right)} \|\Lambda_k\|_{L^p(\mathfrak{v}_n)\to L^q(\mathfrak{v}_n)} \|g\|_{L^p(\mathfrak{v})}$$

*for all $g$ in $\mathcal{S}(\mathfrak{v})$ and all $\rho > 0$.*

**Proof.** Changing variables in the integrals, we obtain

$$\begin{aligned}\|\Pi_k^{\rho\omega} g\|_{L^q(\mathfrak{v})} &= \|\Lambda_k^{\rho}(g_\omega)\circ A_\omega\|_{L^q(\mathfrak{v})} = |\det A_\omega|^{-\frac{1}{q}} \|\Lambda_k^{\rho}(g_\omega)\|_{L^q(\mathfrak{v}_n)} \\ &\le |\det A_\omega|^{-\frac{1}{q}} \|\Lambda_k^{\rho}\|_{L^p(\mathfrak{v}_n)\to L^q(\mathfrak{v}_n)} \|g_\omega\|_{L^p(\mathfrak{v}_n)} \\ &= |\det A_\omega|^{\frac{1}{p}-\frac{1}{q}} \|\Lambda_k^{\rho}\|_{L^p(\mathfrak{v}_n)\to L^q(\mathfrak{v}_n)} \|g\|_{L^p(\mathfrak{v})}.\end{aligned}$$

From (4.16) it then follows that

$$\|\Pi_k^{\rho\omega}\|_{L^p(\mathfrak{v})\to L^q(\mathfrak{v})} \le C\|\Lambda_k^{\rho}\|_{L^p(\mathfrak{v}_n)\to L^q(\mathfrak{v}_n)},$$

which implies the statement by Lemma 2.2. □

We are now ready to work out the spectral resolution of the operator $L$. As in Section 2, to simplify the notation we set $\mu_k := \mu/(2k+n)$.

**Theorem 4.8.** *If $f$ is a Schwartz function on $G$, then*

$$f(V, Z) = \int_0^{\infty} \mathcal{P}_\mu^L f(V, Z)\, d\mu,$$

*where*

$$\mathcal{P}_\mu^L f(V, Z) = \mu^{n+d-1} \sum_{k=0}^{\infty} (2k+n)^{-n-d} \left( \int_S \widehat{h_{\mu_k}}(\omega) \big(\Pi_k^{\mu_k\omega} g\big)(V) e^{i\mu_k\omega(Z)} d\sigma(\omega) \right) \tag{4.19}$$

*and*

$$L\big(\mathcal{P}_\mu^L f\big) = \mu\, \mathcal{P}_\mu^L f.$$

**Proof.** For the sake of simplicity, we suppose that $f$ is a tensor function, that is $f(V, Z) = g(V)\, h(Z)$ with $g$ and $h$ Schwartz functions.

If $V \in \mathfrak{v}$, we have

$$\begin{aligned}\big(\mathfrak{F}_{\mathfrak{z}} f\big)(V, \rho\omega) &= g(V) \int_{\mathfrak{z}} e^{-i\rho\,\omega(Z)} h(Z)\, dZ \\ &= g(V)\widehat{h}(\rho\omega) = g(V)\widehat{h_\rho}(\omega),\end{aligned}$$

where $\widehat{h}$ is a simpler notation for the Fourier transform $\mathfrak{F}_{\mathfrak{z}} h$ of $h$ and $h_\rho(Z) = \frac{1}{\rho^d} h\big(\frac{Z}{\rho}\big)$.

From the expansion (4.18) we then get

$$\big(\mathfrak{F}_{\mathfrak{z}} f\big)(V, \rho\omega) = \rho^n \sum_{k=0}^{\infty} \widehat{h_\rho}(\omega) \big(\Pi_k^{\rho\omega} g\big)(V).$$

Plugging this expression into the inversion formula for the Fourier transform on $\mathfrak{z}$, written in polar coordinates, we obtain

$$f(V,Z) = \frac{1}{(2\pi)^d}\int_0^\infty \left(\int_S e^{i\rho\omega(Z)}\big(\mathfrak{F}_{\mathfrak{z}} f\big)(V,\rho\omega)d\sigma(\omega)\right)\rho^{d-1}d\rho$$
$$= \frac{1}{(2\pi)^d}\sum_{k=0}^\infty \int_0^\infty \left(\int_S \widehat{h_\rho}(\omega)\big(\Pi_k^{\rho\omega} g\big)(V)e^{i\rho\omega(Z)}d\sigma(\omega)\right)\rho^{n+d-1}d\rho. \tag{4.20}$$

Since

$$\Delta_{\mathfrak{z}}\Big(e^{i\rho\omega(Z)}\big(\Pi_k^{\rho\omega} g\big)(V)\Big) = \rho^2 e^{i\rho\omega(Z)}\big(\Pi_k^{\rho\omega} g\big)(V),$$

we see that

$$\Delta_{\mathfrak{z}}\left(\int_S \widehat{h_\rho}(\omega)e^{i\rho\omega(Z)}\big(\Pi_k^{\rho\omega} g\big)(V)d\sigma(\omega)\right) = \rho^2\int_S \widehat{h_\rho}(\omega)e^{i\rho\omega(Z)}\big(\Pi_k^{\rho\omega} g\big)(V)d\sigma(\omega).$$

Similarly, since

$$L\Big(e^{i\rho\omega(Z)}\big(\Pi_k^{\rho\omega} g\big)(V)\Big) = \rho(2k+n)\left(e^{i\rho\omega(Z)}\big(\Pi_k^{\rho\omega} g\big)(V)\right),$$

we have

$$L\left(\int_S \widehat{h_\rho}(\omega)e^{i\rho\omega(Z)}\big(\Pi_k^{\rho\omega} g\big)(V)d\sigma(\omega)\right)$$
$$= \rho(2k+n)\left(\int_S \widehat{h_\rho}(\omega)e^{i\rho\omega(Z)}\big(\Pi_k^{\rho\omega} g\big)(V)d\sigma(\omega)\right).$$

In the sum (4.20) replacing the eigenvalue $\rho$ of $\sqrt{\Delta_{\mathfrak{z}}}$ by the eigenvalue $\mu = \rho(2k+n)$ of $L$ in the integrals, we obtain, as in the case of the Heisenberg group, the spectral decomposition of $L$,

$$f(V,Z) = \sum_{k=0}^\infty (2k+n)^{-n-d}\int_0^\infty \mu^{n+d-1}\left(\int_S \widehat{h_{\mu_k}}(\omega)\Big(\Pi_k^{\mu_k\omega} g\Big)(V)e^{i\mu_k\omega(Z)}d\sigma(\omega)\right)d\mu.$$

In particular, being

$$L\left(\int_S \widehat{h_{\mu_k}}(\omega)\big(\Pi_k^{\mu_k\omega} g\big)(V)e^{i\mu_k\omega(Z)}d\sigma(\omega)\right)$$
$$= \mu\left(\int_S \widehat{h_{\mu_k}}(\omega)\big(\Pi_k^{\mu_k\omega} g\big)(V)e^{i\mu_k\omega(Z)}d\sigma(\omega)\right),$$

if we define $\mathcal{P}_\mu^L$ as in (4.19), we see that

$$L\big(\mathcal{P}_\mu^L f\big) = \mu\,\mathcal{P}_\mu^L f$$

and

$$f(V,Z) = \int_0^\infty \mathcal{P}_\mu^L f(V,Z)d\mu. \quad \square$$

## 5. Restriction estimates for Métivier groups

In this section we estimate the norms of the operators $\mathcal{P}^L_\mu$ in the general framework of a Métivier group $G$. As for the Heisenberg group, we first prove a conditional result, guaranteeing the boundedness of $\mathcal{P}^L_\mu$ from $L^p(G)$ to $L^q(G)$ on the assumption that the projections $\Lambda_k$ of the twisted Laplacian are bounded from $L^p(\mathfrak{v}_n)$ to $L^q(\mathfrak{v}_n)$.

**Theorem 5.1.** *Assume that $1 \le r \le 2\frac{d+1}{d+3}$. If the projections $\Lambda_k$ are bounded from $L^p(\mathfrak{v}_n)$ to $L^q(\mathfrak{v}_n)$, with $1 \le p \le 2 \le q \le \infty$, then the following inequality holds*

$$\left\| \mathcal{P}^L_\mu f \right\|_{L^{r'}(\mathfrak{z})L^q(\mathfrak{v})} \le C\mu^{d\left(\frac{1}{r}-\frac{1}{r'}\right)+n\left(\frac{1}{p}-\frac{1}{q}\right)-1} \left( \sum_{k=0}^{\infty} (2k+n)^{-d\left(\frac{1}{r}-\frac{1}{r'}\right)-n\left(\frac{1}{p}-\frac{1}{q}\right)} \right.$$
$$\left. \times \left\| \Lambda_k \right\|_{L^p(\mathfrak{v}_n)\to L^q(\mathfrak{v}_n)} \right) \| f \|_{L^r(\mathfrak{z})L^p(\mathfrak{v})}. \tag{5.1}$$

**Proof.** In order to simplify the notation, we write $f(V, Z) = h(Z)g(V)$, with $h$ and $g$ Schwartz functions. However, in the proof we will never use this fact. We take $\alpha : \mathfrak{v} \to \mathbb{C}$ and $\beta : \mathfrak{z} \to \mathbb{C}$, $\alpha \in \mathcal{S}(\mathfrak{v})$, $\beta \in \mathcal{S}(\mathfrak{z})$. Then

$$\begin{aligned}
\langle \mathcal{P}^L_\mu f, \alpha \otimes \beta \rangle_{\mathfrak{v}\oplus\mathfrak{z}} &= \int_{\mathfrak{v}} \int_{\mathfrak{z}} \overline{\alpha(V)\beta(Z)} \mu^{n+d-1} \sum_{k=0}^{\infty} (2k+n)^{-n-d} \\
&\quad \times \left( \int_S e^{i\mu_k\omega(Z)} \widehat{h_{\mu_k}}(\omega) \left( \Pi_k^{\mu_k\omega} g \right)(V) d\sigma(\omega) \right) dZ\, dV \\
&= \mu^{n+d-1} \sum_{k=0}^{\infty} (2k+n)^{-n-d} \left( \int_S \int_{\mathfrak{v}} \widehat{h_{\mu_k}}(\omega) \left( \Pi_k^{\mu_k\omega} g \right)(V) \right. \\
&\quad \left. \times \int_{\mathfrak{z}} e^{i\mu_k\omega(Z)} \overline{\alpha(V)\beta(Z)} dZ dV d\sigma(\omega) \right) \\
&= \mu^{n+d-1} \sum_{k=0}^{\infty} (2k+n)^{-n-d} \\
&\quad \times \left( \int_S \left\langle \widehat{h_{\mu_k}}(\omega) \left( \Pi_k^{\mu_k\omega} g \right), \widehat{\beta_{\mu_k}}(\omega)\alpha \right\rangle_{\mathfrak{v}} d\sigma(\omega) \right).
\end{aligned}$$

Applying Hölder's inequality to the inner integral we deduce that

$$\begin{aligned}
\left| \langle \mathcal{P}^L_\mu f, \alpha \otimes \beta \rangle_{\mathfrak{v}\oplus\mathfrak{z}} \right| &\le \mu^{n+d-1} \sum_{k=0}^{\infty} (2k+n)^{-n-d} \\
&\quad \times \left( \int_S \left\| \widehat{h_{\mu_k}}(\omega) \left( \Pi_k^{\mu_k\omega} g \right) \right\|_{L^q(\mathfrak{v})} \| \widehat{\beta_{\mu_k}}(\omega)\alpha \|_{L^{q'}(\mathfrak{v})} d\sigma(\omega) \right).
\end{aligned}$$

Using Lemma 4.7, we then obtain

$$\begin{aligned}
\left|\langle \mathcal{P}^L_\mu f, \alpha \otimes \beta\rangle_{\mathfrak{v}\oplus\mathfrak{z}}\right| &\le C\mu^{n+d-1}\sum_{k=0}^{\infty}(2k+n)^{-n-d}\,\|\Lambda_k\|_{L^p(\mathfrak{v}_n)\to L^q(\mathfrak{v}_n)}\,\lambda_k(\mu)^{n\left(\frac{1}{p}-\frac{1}{q}-1\right)}\\
&\quad\times\left(\int_S \|\widehat{h_{\mu_k}}(\omega)g\|_{L^p(\mathfrak{v})}\|\widehat{\beta_{\mu_k}}(\omega)\alpha\|_{L^{q'}(\mathfrak{v})}d\sigma(\omega)\right)\\
&\le C\mu^{d+n\left(\frac{1}{p}-\frac{1}{q}\right)-1}\sum_{k=0}^{\infty}(2k+n)^{-n\left(\frac{1}{p}-\frac{1}{q}\right)-d}\,\|\Lambda_k\|_{L^p(\mathfrak{v}_n)\to L^q(\mathfrak{v}_n)}\\
&\quad\times\left(\int_S \|\widehat{h_{\mu_k}}(\omega)g\|_{L^p(\mathfrak{v})}\|\widehat{\beta_{\mu_k}}(\omega)\alpha\|_{L^{q'}(\mathfrak{v})}d\sigma(\omega)\right).
\end{aligned}$$

Then the Cauchy–Schwarz inequality implies

$$\begin{aligned}
\left|\langle \mathcal{P}^L_\mu f, \alpha \otimes \beta\rangle_{\mathfrak{v}\oplus\mathfrak{z}}\right| &\le C\mu^{d+n\left(\frac{1}{p}-\frac{1}{q}\right)-1}\sum_{k=0}^{\infty}(2k+n)^{-d-n\left(\frac{1}{p}-\frac{1}{q}\right)}\|\Lambda_k\|_{L^p(\mathfrak{v}_n)\to L^q(\mathfrak{v}_n)}\\
&\quad\times\left(\int_S \|\widehat{h_{\mu_k}}(\omega)g\|^2_{L^p(\mathfrak{v})}d\sigma(\omega)\right)^{\frac{1}{2}}\left(\int_S \|\widehat{\beta_{\mu_k}}(\omega)\alpha\|^2_{L^{q'}(\mathfrak{v})}d\sigma(\omega)\right)^{\frac{1}{2}}.
\end{aligned}$$

Since $p \le 2 \le q$, it follows that $\frac{2}{p} \ge 1$ and $\frac{2}{q'} \ge 1$. Therefore we can apply to the integrals on the right hand side the Minkowski integral inequality, to attain

$$\begin{aligned}
\left|\langle \mathcal{P}^L_\mu f, \alpha \otimes \beta\rangle_{\mathfrak{v}\oplus\mathfrak{z}}\right| &\le C\mu^{d+n\left(\frac{1}{p}-\frac{1}{q}\right)-1}\sum_{k=0}^{\infty}(2k+n)^{-d-n\left(\frac{1}{p}-\frac{1}{q}\right)}\|\Lambda_k\|_{L^p(\mathfrak{v}_n)\to L^q(\mathfrak{v}_n)}\\
&\quad\times\left(\int_S\left(\int_{\mathfrak{v}}\left|\widehat{h_{\mu_k}}(\omega)g(V)\right|^p dV\right)^{\frac{2}{p}}d\sigma(\omega)\right)^{\frac{1}{2}}\\
&\quad\times\left(\int_S\left(\int_{\mathfrak{v}}\left|\widehat{\beta_{\mu_k}}(\omega)\alpha(V)\right|^{q'} dV\right)^{\frac{2}{q'}}d\sigma(\omega)\right)^{\frac{1}{2}}\\
&\le C\mu^{d+n\left(\frac{1}{p}-\frac{1}{q}\right)-1}\sum_{k=0}^{\infty}(2k+n)^{-d-n\left(\frac{1}{p}-\frac{1}{q}\right)}\|\Lambda_k\|_{L^p(\mathfrak{v}_n)\to L^q(\mathfrak{v}_n)}\\
&\quad\times\left(\int_{\mathfrak{v}}\left(\int_S\left|\widehat{h_{\mu_k}}(\omega)g(V)\right|^2 d\sigma(\omega)\right)^{\frac{p}{2}}dV\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_{\mathfrak{v}}\left(\int_S\left|\widehat{\beta_{\mu_k}}(\omega)\alpha(V)\right|^2 d\sigma(\omega)\right)^{\frac{q'}{2}}dV\right)^{\frac{1}{q'}}.
\end{aligned}$$

The Stein–Tomas theorem, that for $1 \le r \le p_*(d)$ furnishes the bound

$$\|\widehat{h_\mu}\|_{L^2(S)} \le C\|h_\mu\|_{L^r(\mathfrak{z})} = C\mu^{-\frac{d}{r'}}\|h\|_{L^r(\mathfrak{z})},$$

may now be pressed into service to give

$$\begin{aligned}\left|\langle\mathcal{P}_\mu^L f,\alpha\otimes\beta\rangle_{\mathfrak{v}\oplus\mathfrak{z}}\right| &\le C\mu^{d-1+n\left(\frac{1}{p}-\frac{1}{q}\right)}\mu^{-2\frac{d}{r'}}\\ &\quad\times\left(\sum_{k=0}^{\infty}(2k+n)^{-\left(d+n\left(\frac{1}{p}-\frac{1}{q}\right)-2\frac{d}{r'}\right)}\left\|\Lambda_k\right\|_{L^p(\mathfrak{v}_n)\to L^q(\mathfrak{v}_n)}\right)\\ &\quad\times\left(\int_{\mathfrak{v}}\left(\int_{\mathfrak{z}}\left|h(Z)g(V)\right|^r dZ\right)^{\frac{p}{r}}dV\right)^{\frac{1}{p}}\\ &\quad\times\left(\int_{\mathfrak{v}}\left(\int_{\mathfrak{z}}\left|\beta(Z)\alpha(V)\right|^r dZ\right)^{\frac{q'}{r}}dV\right)^{\frac{1}{q'}},\end{aligned}$$

whence it follows that

$$\begin{aligned}\left\|\mathcal{P}_\mu^L f\right\|_{L^{r'}(\mathfrak{z})L^q(\mathfrak{v})} &\le C\mu^{d\left(\frac{1}{r}-\frac{1}{r'}\right)+n\left(\frac{1}{p}-\frac{1}{q}\right)-1}\left\|f\right\|_{L^r(\mathfrak{z})L^p(\mathfrak{v})}\\ &\quad\times\left(\sum_{k=0}^{\infty}(2k+n)^{-d\left(\frac{1}{r}-\frac{1}{r'}\right)-n\left(\frac{1}{p}-\frac{1}{q}\right)}\left\|\Lambda_k\right\|_{L^p(\mathfrak{v}_n)\to L^q(\mathfrak{v}_n)}\right),\end{aligned}$$

proving the assertion. □

Implementing Theorem 5.1 with the Koch–Ricci estimates for $\left\|\Lambda_k\right\|_{L^p(\mathfrak{v}_n)\to L^q(\mathfrak{v}_n)}$, we are finally able to prove Theorem 1.3.

**Proof of Theorem 1.3.** To prove (1.5) we only need to discuss the convergence of the series in (5.1),

$$\sum_{k=0}^{\infty}(2k+n)^{-d\left(\frac{1}{r}-\frac{1}{r'}\right)-n\left(\frac{1}{p}-\frac{1}{q}\right)}\left\|\Lambda_k\right\|_{L^p(\mathfrak{v}_n)\to L^q(\mathfrak{v}_n)}.$$

Comparing this sum with (2.16), the corresponding one in the case of the Heisenberg group, we see that here we have the factor $(2k+n)^{-d(\frac{1}{r}-\frac{1}{r'})}$ in place of $(2k+n)^{-1}$. Since $d(\frac{1}{r}-\frac{1}{r'})>1$ for $1\le r\le 2\frac{d+1}{d+3}$ and $d\ge 2$, the convergence is improved. This observation alone suffices to prove the statement. Indeed, when $(p,q)\ne(2,2)$ the corresponding series for the Heisenberg group converges in the prescribed range and, when $(p,q)=(2,2)$, our series converges since $d(\frac{1}{r}-\frac{1}{r'})>1$, as previously observed. □

**Remark 5.2.** The following example shows that the range of $r$ cannot be extended. It is manufactured by mixing the Knapp counterexample to the Stein–Tomas theorem and Müller's example, showing that estimates between Lebesgue spaces for the operators $\mathcal{P}_\mu$ are necessarily trivial. Let $h$ be a Schwartz function on $\mathfrak{z}$ and $g(V)=\varphi_0\left(\frac{V}{\sqrt{n}}\right)=e^{-\frac{|V|^2}{4n}}$. Define $f(V,Z)=g(V)h(Z)$, then

$$\mathcal{P}_1^L f(V,Z)=\frac{1}{n^{n+1}}g(V)\int_S\widehat{h}(\omega)\,e^{-i\omega(Z)}d\omega=\frac{1}{n^{n+1}}g(V)(R^*Rh)(Z),$$

where $R$ and $R^*$ are the restriction and extension operators. Hence, an estimate

$$\|\mathcal{P}_1^L f\|_{L^{r'}(\mathfrak{z})L^q(\mathfrak{v})}\le C\|f\|_{L^r(\mathfrak{z})L^p(\mathfrak{v})}$$

with $2\frac{d+1}{d+3} < r$ would violate the sharpness of the Stein–Tomas theorem which is guaranteed by the Knapp example, since it would imply

$$\|R^*Rh\|_{L^{r'}(\mathfrak{z})} \le C\|h\|_{L^r(\mathfrak{z})},$$

for all Schwartz functions $h$ on $\mathfrak{z} = \mathbb{R}^d$.

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